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THE GENERATION OF BEAMS OF THREE-DIMENSIONAL PERIODIC INTERNAL WAVES IN AN EXPONENTIALLY STRATIFIED FLUID†

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The spectral method is used to construct an exact solution of the linearized problem of the generation of disturbances by localized sources that execute arbitrary periodic motions in a viscous exponentially stratified fluid. The expressions obtained do not contain any adjusting parameters and describe conical beams of three-dimensional periodic internal waves and two types of boundary layers, the spatial scale of which is given by the kinematic viscosity and the buoyancy frequency of the medium. The thickness of one of them, which is analogous to Stokes periodic flow in a homogeneous viscous fluid, is specified by the kinematic viscosity and the wave frequency, that is, it additionally depends on a ratio of the wave and buoyancy frequencies. The thickness of the specific internal boundary layer also depends on the geometry of the problem. In the approximation of weak stratification and low viscosity, asymptotic estimates of the expressions obtained are presented for two types of generators, namely, in the form of a plane inclined rectangle that vibrates along its surface (a frictional source) and along the normal to it (a piston source) in the non-degenerate case when the wave cone does not touch the radiating plane. In limiting cases the analytical expressions obtained agree with known exact solutions of the problem of generating axially symmetric and two-dimensional periodic internal waves. © 2003 Elsevier Ltd. All rights reserved.

An approximate solution of the problem of the generation of two-dimensional infinitesimal periodic waves in a viscous stratified fluid by a horizontal elliptic cylinder is known [1]. The exact solution of the problem of the excitation of internal waves by a vibrating inclined strip takes into account the boundary layer that arises [2]. In practice the sources of periodic waves in most cases are localized, and the internal waves are three-dimensional ones [3]. In the theory of wave processes such three-dimensional periodic waves propagating along the generatrix of wave cones play an essential role. A calculation of three-dimensional harmonic internal waves has been carried out only for a single special case, when the generator is part of the surface of a vertical cylinder [4], and the calculations are simplified considerably due to the match between the symmetries of the radiator and the wave-field geometry.

The purpose of the present paper is to construct a solution of the linearized three-dimensional problem of the generation of a set of disturbances in a viscous exponentially stratified fluid by part of an inclined plane that executes arbitrary periodic motions. A complete solution that satisfies the system of equations of motion and the exact boundary conditions is found using the method presented previously in [2, 4].

1. THE EQUATIONS OF MOTION AND THE BOUNDARY CONDITIONS

Consider a stratified fluid whose density decays exponentially with height z: $\rho_0(z) = \rho_{00}\exp(-z/\Lambda)$, where Λ is the buoyancy scale and the z axis is directed opposite to the direction of the acceleration due to gravity $\mathbf{g}$. In the Bussinesq approximation the linearized system of the equations of motion of a viscous incompressible fluid when there is no diffusion of salt has the form [5]

$$\rho_0\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial P}{\partial z} + \rho_0 \nu\Delta\mathbf{v} - \rho g\mathbf{e}_z, \quad \frac{\partial\rho}{\partial t} - \upsilon_z\frac{\rho_0}{\Lambda} = 0, \quad \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

where $\mathbf{v} = (\upsilon_x, \upsilon_y, \upsilon_z)$, ρ and P are the varying velocity, density and pressure respectively, and $\mathbf{e}_z$ is the unit vector directed along the z axis. Under natural and laboratory conditions the stratification is usually weak ($\Lambda \gg H$, where H is the maximum linear scale of the problem), and the viscosity is low ($N\lambda_c^2 \gg \nu$, where λ_c is the characteristic wavelength and ν is the kinematic viscosity). These properties are widely used in the theory of internal waves [1–4].

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The no-slip conditions on the boundary surface Σ are the boundary conditions for system of equations (1.1). In a viscous medium all disturbances decay at infinity. The medium is initially at rest. Under steady conditions of oscillation, which are considered in what follows, the time-dependence of all quantities is harmonic, and so we will everywhere omit the common factor $\exp(-i\omega t)$.

Part of an infinite plane, oriented at an arbitrary angle φ with respect to the horizon, and which executes periodic motions with frequency ω and velocity amplitude u_0, serves as the source of waves. The displacements occur in arbitrary directions, including the direction parallel to the surface (similar to the two-dimensional case [2]) or normal to it.

To simplify the writing, we will use simultaneously several orthogonal systems of coordinates, which are shown in the figure. The line of action of the gravity force defines the natural laboratory system of coordinates (x, y, z).

The local system of coordinates (ξ, η, ζ) is connected with the radiating surface, which lies in the $O\xi\eta$ plane and without loss of generality can be obtained by rotating the initial system of coordinates (x, y, z) by an angle φ around the y axis. With this choice, the ξ, η axes lie in the plane of the source, and the ζ axis is normal to it.

The accompanying system of coordinates (q, p, α) is connected with the wave cone, the q axis is directed along the wave cone, the p axis is directed transversely, and α is the angular variable. Connections between these systems and the auxiliary cylindrical system (r, α, z) are given by the relations

$$\begin{aligned} &\xi = x\cos\varphi + z\sin\varphi, \quad \eta = y, \quad \zeta = -x\sin\varphi + z\cos\varphi \\ &x = r\cos\alpha, \quad y = r\sin\alpha \quad z = z \\ &p = r\sin\theta - z\cos\theta, \quad q = r\cos\theta + z\sin\theta \end{aligned} \tag{1.2}$$

The form of the incompressibility condition in system (1.1) enables one to introduce the toroidal-poloidal decomposition [6], that defines the three-dimensional fluid velocity in terms of two auxiliary scalar functions Φ and Ψ

$$\mathbf{v} = \nabla\times\mathbf{e}_z\Psi + \nabla\times(\nabla\times\mathbf{e}_z\Phi) \tag{1.3}$$

The introduction of the potentials Φ and Ψ, although it also leads to an increase in the order of the system of equations (1.1), considerably simplifies the calculations, since it enables one to eliminate mixed derivatives with respect to the coordinates.

Eliminating the pressure from system (1.1) and using expression (1.3), we obtain two equations for finding the functions Φ and Ψ

$$(\omega^2\Delta - N^2\Delta_\perp - i\omega\nu\Delta^2)\Delta_\perp\Phi = 0 \quad (\omega - i\nu\Delta)\Delta_\perp\Psi = 0 \tag{1.4}$$

where Δ is the Laplace operator, $\Delta_\perp = \partial_{xx}^2 + \partial_{yy}^2$ and $N = \sqrt{g/\Lambda}$ is the buoyancy frequency. In view of expression (1.3), the solution of the equation $\Delta_\perp\Phi = 0$ corresponds to isopycnic motion of the fluid with zero vertical component of the velocity. Since such motion is also determined by the poloidal part of the potential Ψ, the solution of the equation $\Delta_\perp\Phi = 0$ cannot be taken into account when analysing the wave disturbances caused by particle displacements from the horizon of neutral buoyancy.

Besides, the solution of the equation $\Delta_\perp\Phi = 0$ (the second equations of (1.4)) describes non-dissipative motion of the fluid. However the equations of system (1.1) include a horizontal component of the friction force $\rho_0\nu(\partial^2\mathbf{v}/\partial z^2)$, which is equal to zero only for motions of the form $\mathbf{v} = \mathbf{a}(x, y) + z\mathbf{b}(x, y)$. Since such motions cannot be excited by a source of finite size, they will not be considered in what follows.

Taking the points outlined above into account, system (1.1) can be written in the form

$$(\omega^2\Delta - N^2\Delta_\perp - i\omega\nu\Delta^2)\Phi = 0 \quad (\omega - i\nu\Delta)\Psi = 0 \tag{1.5}$$

Disturbances of the pressure and the density are defined by the expressions

$$P = \rho_0(i\omega + \nu\Delta)\partial\Phi/\partial z, \quad \rho = -i\rho_0\Delta_\perp\Phi/\omega\Lambda$$

To simplify the writing, system (1.5) and the boundary conditions are presented in the local system of coordinates (ξ, η, ζ), connected with the radiating surface (see the figure).

$$\begin{aligned} &\omega^2\Delta\Phi - N^2\left\lfloor(\cos\varphi\partial_\xi - \sin\varphi\partial_\zeta)^2 + \partial_\eta^2\right\rfloor\Phi - i\omega\nu\Delta^2\Phi = 0 \\ &(\omega - i\nu\Delta)\Psi = 0 \end{aligned} \tag{1.6}$$

On the moving part of the plane we have $\mathbf{v}|_{\zeta=0} = \mathbf{u}(\xi, \eta)$, and the boundary conditions for the potentials take the form

$$\begin{aligned}
&\cos\varphi\partial_\eta\Psi + \lfloor -\sin\varphi(\partial_\eta^2 + \partial_\xi^2) + \cos\varphi\partial_{\xi\zeta}^2 \rfloor\Phi|_{\zeta=0} = u_\xi(\xi, \eta) \\
&-(\cos\varphi\partial_\xi - \sin\varphi\partial_\eta)\Psi + \partial_\eta(\sin\varphi\partial_\xi + \cos\varphi\partial_\zeta)\Phi|_{\zeta=0} = u_\eta(\xi, \eta) \\
&-\sin\varphi\partial_\eta\Psi + \lfloor -\cos\varphi(\partial_\eta^2 + \partial_\xi^2) + \sin\varphi\partial_{\xi\zeta}^2 \rfloor\Phi|_{\zeta=0} = u_\zeta(\xi, \eta)
\end{aligned} \qquad (1.7)$$

where $\Phi, \Psi \to 0$ as $\xi, \eta, \zeta \to \infty$.

Solutions of system (1.6), will be sought in the form of an expansion in Fourier integrals over the whole space. Due to the symmetry of the wave field, the analysis will be carried out only for the upper half space ($\zeta > 0$).

2. CONSTRUCTION OF THE SOLUTION

In the theory of periodic internal waves (ω = const) the wave vectors $\mathbf{k} = (k_x, k_y, k_z)$ that occur in the expansions for the potentials

$$\begin{aligned}
\Phi &= \int_{-\infty}^{+\infty} [A(k_\xi, k_\eta)\exp(ik_1(k_\xi, k_\eta)\zeta) + B(k_\xi, k_\eta)\exp(ik_2(k_\xi, k_\eta)\zeta)] \times \\
&\times \exp(ik_\xi\xi + ik_\eta\eta)dk_\xi dk_\eta \\
\Psi &= \int_{-\infty}^{+\infty} C(k_\xi, k_\eta)\exp(ik_3(k_\xi, k_\eta)\zeta + ik_\xi\xi + ik_\eta\eta)dk_\xi dk_\eta
\end{aligned} \qquad (2.1)$$

are defined by the solutions of the dispersion equation corresponding to system (1.1) and are expressed in terms of two components of the wave vector $k_i = k_i(k_\xi, k_\eta)$ $(i = 1, 2, 3)$. Direct substitution of the expressions for the potentials (2.1) into system (1.6) gives the following equations for finding all components of the vector $\mathbf{k}$

$$\begin{aligned}
&\omega^2(k_{1,2}^2 + k_\xi^2 + k_\eta^2) - N^2[(k_\xi\cos\varphi - k_{1,2}\sin\varphi)^2 + k_\eta^2] + i\omega\nu(k_{1,2}^2 + k_\xi^2 + k_\eta^2)^2 = 0 \\
&k_3^2 = -\frac{\omega}{i\nu} - k_\xi^2 - k_\eta^2
\end{aligned} \qquad (2.2)$$

The conditions for the disturbances to decay at infinity are satisfied when the imaginary parts of the roots of the dispersion equation are greater than zero: $\mathrm{Im}k_1 > 0$, $\mathrm{Im}k_2 > 0$ and $\mathrm{Im}k_3 > 0$.

The coefficients A, B and C are found from the solutions of the system of differential equations obtained after substituting expressions (2.1) into boundary conditions (1.8)

$$\begin{cases}
A(k_\eta^2\sin\varphi + k_1\beta_1) + B(k_\eta^2\sin\varphi + k_2\beta_2) + iCk_\eta\cos\varphi = U_\xi \\
-Ak_\eta\gamma_1 - Bk_\eta\gamma_2 + iC\gamma_3 = U_\eta \\
A(k_\eta^2\cos\varphi - k_\xi\beta_1) + B(k_\eta^2\cos\varphi - k_\xi\beta_2) - iCk_\eta\sin\varphi = U_\zeta
\end{cases} \qquad (2.3)$$

where

$$\begin{aligned}
&\mathbf{U} = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \mathbf{u}(\xi, \eta)\exp(-ik_\xi\xi - ik_\eta\eta)d\xi d\eta \\
&\beta_i = k_i\sin\varphi - k_\xi\cos\varphi, \quad \gamma_i = k_i\cos\varphi + k_\xi\sin\varphi
\end{aligned} \qquad (2.4)$$

($\mathbf{U}(k_\xi, k_\eta)$ is the Fourier transform of the source velocity $\mathbf{u}(\xi, \eta)$).

The solutions of system (2.4) can be represented in the matrix form

$$\begin{Vmatrix} A \\ B \\ C \end{Vmatrix} = \frac{1}{\Delta}\begin{Vmatrix} D_{11} & D_{12} & D_{31} \\ D_{21} & D_{22} & D_{32} \\ D_{31} & D_{32} & D_{33} \end{Vmatrix}\begin{Vmatrix} U_\xi \\ U_\eta \\ U_\zeta \end{Vmatrix} \tag{2.5}$$

which reflects the nature of the dispersion of three-dimensional internal waves. The matrix coefficients D_{ij} are defined by the expressions

$$D_{11} = -ik_\eta^2\left[k_\xi + \frac{1}{2}(k_2 - k_3)\sin 2\varphi\right] - ik_\xi\beta_2\beta_3, \quad D_{12} = -ik_\eta(k_\eta^2 + \beta_2^2)$$

$$D_{13} = -ik_\eta^2[(k_2 - k_3)\cos^2\varphi + k_3] - ik_2\beta_2\beta_3$$

$$D_{21} = -ik_\eta^2\left[k_\xi + \frac{1}{2}(k_1 - k_3)\sin 2\varphi\right] - ik_\xi\beta_1\beta_3, \quad D_{22} = -ik_\eta(k_\eta^2 + \beta_1^2)$$

$$D_{23} = -ik_\eta^2[(k_1 - k_3)\cos^2\varphi + k_3] - ik_1\beta_1\beta_3$$

$$D_{31} = (k_\eta^2\cos^2\varphi + k_\xi^2)(k_1 - k_2)k_\eta$$

$$D_{32} = (k_1 - k_2)\left[-k_\xi\beta_1\beta_2 + k_\eta^2\left(\frac{1}{2}(k_1 + k_2)\sin 2\varphi + k_\xi\cos 2\varphi\right)\right]$$

$$D_{33} = (k_1 - k_2)k_\eta\sin\varphi\left[-k_\eta^2\cos\varphi + k_\eta(k_1\gamma_2 + k_\xi\beta_2)\right]$$

$$\Delta = (k_1 - k_2)\{-k_\xi\beta_1\beta_2\beta_3 + ik_\eta^4\cos\varphi + ik_\eta^2\left[\beta_3(\gamma_1\sin\varphi + \beta_2\cos\varphi) + \gamma_1\beta_1^2 - \gamma_2\beta_2^2\right]\}$$

Expressions (2.1), in which the coefficients $A(k_\xi, k_\eta)$, $B(k_\xi, k_\eta)$ and $C(k_\xi, k_\eta)$ are defined by formulas (2.5), are the general solution of system (1.1), which exactly satisfy boundary conditions (1.7). The family of solutions (2.1) includes not only waves, but also accompanying periodic solutions, the properties of which are defined by the relations between the real and imaginary parts of the solutions of dispersion equations (2.2).

3. THE SOLUTION OF THE DISPERSION EQUATIONS

Taking into account the smallness of the viscosity, the solutions of dispersion equations (2.2) are found by standard methods [7]

$$\begin{aligned}
k_1 &= k_1^{(0)} + i\nu k_1^{(1)} \\
k_1^{(0)} &= \frac{k_\xi\sin 2\varphi + 2\kappa\cos\theta}{2\mu}, \quad k_1^{(1)} = \frac{\sin\theta(k_\xi\sin\varphi\cos\theta + \kappa\cos\varphi)^4}{2N\mu^4\kappa\cos\theta} \\
k_2 &= \frac{i + \operatorname{sign}\mu}{\delta_\varphi} - \frac{k_\xi\sin 2\varphi}{2\mu} + k_2^{(1)}\delta_\varphi \\
k_2^{(1)} &= \frac{i + \operatorname{sign}\mu}{6\sqrt{2}}\,\frac{k_\xi^2(\mu(\cos^2\varphi - \sin^2\theta) + 2\sin\varphi\cos\varphi) + k_\eta^2\mu\cos^2\theta}{\mu^2} \\
k_3 &= \frac{i+1}{\delta_\nu} + \frac{i+1}{4}\delta_\nu(k_\xi^2 + k_\eta^2) \\
\kappa &= \sqrt{k_\xi^2\sin^2\theta - \mu k_\eta^2}, \quad \mu = \sin^2\varphi - \sin^2\theta, \quad \theta = \arcsin(\omega/N) \\
\delta_\varphi &= \sqrt{\frac{2\nu\sin\theta}{N|\mu|}}, \quad \delta_\nu = \sqrt{\frac{2\nu}{\omega}}
\end{aligned} \tag{3.1}$$

where θ is the angle which the wave cone makes with the horizontal, δ_φ is the thickness of the internal boundary layer, calculated for the first time in the two-dimensional formulation of the problem [2] and δ_ν is the thickness of the periodic (Stokes') boundary layer, which also exists in a homogeneous viscous fluid [5].

Formulae (3.1) describe two types of motion: large-scale motion with characteristic dimension $\lambda_C = 2\pi/k$ ($\mathrm{Re}k_1 \gg \mathrm{Im}k_1$) and small-scale motion with characteristic thickness $O(\sqrt{\nu/\omega})$ ($\mathrm{Re}k_{2,3} \sim \mathrm{Im}k_{2,3}$).

In the approximation considered the root k_1 differs from the wave number for an ideal fluid only by a small imaginary correction. The real part of k_1 defines the length of the radiated wave, while the imaginary part defines the spatial structure of the beam of waves and the magnitude of the viscous attenuation. Taking into account relations (2.1) and (3.1) we note that the internal waves are characterized by the spectral density $A(k_\xi, k_\eta)$.

The roots k_2 and k_3 describe periodic boundary flows. The expression with the coefficient $B(k_\xi, k_\eta)$ defines the internal-wave boundary layer of thickness δ_φ. This type of periodic motion is a specific feature of a stratified fluid and has no analogues in a homogeneous medium.

The coefficient $C(k_\xi, k_\eta)$ specifies a viscous periodic boundary layer of thickness δ_ν, which exists both in a stratified fluid and in a homogeneous fluid [5], where, in the three-dimensional case, it is degenerate (the multiple root in the corresponding solution of the dispersion equation).

In the case of a homogeneous fluid ($N \to 0$ and ν is finite) the roots of the dispersion equations have the form

$$k_1 = i\sqrt{k_\xi^2 + k_\eta^2}, \quad k_2 = k_3 = \sqrt{i\omega/\nu - k_\xi^2 - k_\eta^2}$$

The merging of the roots k_2 and k_3 in the exact solution on changing to a homogeneous fluid indicates that the problem is degenerate. The inverse transition from a homogeneous fluid to a stratified fluid cannot be carried out in a uniform manner since in this case the degenerate periodic wave boundary layer splits into two others with a different dependence of their characteristics on parameters of the problem.

On changing to a homogeneous fluid the approximate solutions of dispersion equations (3.1) take the form $k_2 = -k_\xi \sin 2\varphi/(2\mu)$ and $k_3 = 0$. The differences in the structure of the expressions derived are a consequence of the use of regular and asymptotic methods of solving the dispersion equations. The paradox of "critical angles" in the theory of internal waves, which disappears if one uses the regular methods over the whole range of angular parameters of the problem (θ, φ) [8], is connected with this.

4. ANALYSIS OF THE GENERAL SOLUTION IN THE LOW-VISCOSITY APPROXIMATION

The expressions for the coefficients A, B and C in (2.6) are simplified considerably in the case of low viscosity, and they take the form

$$A(k_\xi, k_\eta) \approx \frac{1 - i\,\mathrm{sign}\mu}{2}\frac{k_\xi}{\chi}\delta_\varphi^2 U_\xi + \frac{1-i}{2}\frac{k_\eta}{\chi}\delta_\nu^2 U_\eta - \frac{1}{\chi}U_\zeta \tag{4.1}$$

$$B(k_\xi, k_\eta) \approx \frac{i\sin\theta}{|\mu|\sin\varphi}\delta_N^2 U_\xi + \frac{1+i}{\sqrt{2}}\frac{\sqrt{\sin\theta}}{|\mu|\sin^2\varphi}\frac{k_\eta(k_\eta^2 + \sigma_1^2)}{\chi}\delta_N^3 U_\eta + \frac{i\sin\theta}{|\mu|\sin\varphi}\frac{k_\eta^2\sin\varphi + k_1^{(0)}\sigma_1}{\chi}\delta_N^2 U_\zeta \tag{4.2}$$

$$C(k_\xi, k_\eta) \approx -\frac{i + \mathrm{sign}\mu}{1+i}\frac{|\mu|^{3/2}}{\sin^2\varphi}\frac{k_\eta(k_\eta^2\cos^2\varphi + k_\xi^2)}{\chi}\delta_N^2 U_\xi - \frac{i+1}{\sqrt{2}}\frac{\mathrm{sign}\mu}{\sin\varphi}\delta_N U_\eta - \frac{i+1}{\sqrt{2}}\frac{\mathrm{sign}\mu}{\sin\varphi}\frac{k_\eta(k_1^{(0)}\cos\varphi + k_\xi\sin\varphi)}{\chi}\delta_N U_\zeta \tag{4.3}$$

where

$$\chi = k_\eta^2\cos\varphi - k_\xi\sigma_1$$

$$\sigma_1 = k_1^{(0)}\sin\varphi - k_\xi\cos\varphi = (k_\xi\cos\varphi\sin^2\theta + \kappa\cos\theta\sin\varphi)/\mu$$

In explicit form the coefficients A, B and C are given by their spectral representations U_ξ, U_η and U_ζ, that is, by the form of the source and the mature of its motion.

In the expressions presented above the only parameter with the dimension of length is the universal microscale of periodic motions $\delta_N = \sqrt{\nu/N}$, that is the characteristic internal scale of the problem, which is determined only by the properties of the medium under consideration, namely, by the kinematic viscosity and the buoyancy frequency. Since this parameter is present in all expressions (4.1)–(4.4), it is fundamental in the theory of internal waves in a viscous fluid. In practice this scale must be distinguished in the structure of the spatial spectra of all types of motion in stably stratified media in which the wave component is always present.

To obtain expressions that allow a direct comparison with laboratory experiment, as the radiator of internal waves we consider a rectangle with sides a and b, and, by analogy with the approach proposed previously [4], we analyse two types of generators of internal waves, namely, a *frictional* generator when the plane vibrates along its surface in the (ξ, η) plane, and a piston generator when the displacement vector is normal to the plane of the source.

5. WAVE FIELDS FROM VARIOUS SOURCES

Assuming, without loss of generality, that the rectangle moves only along the longitudinal ξ axis (see the figure), that is

$$\mathbf{u} = u\mathbf{e}_\xi, \quad u = u_0\vartheta(a/2 - |\xi|)\vartheta(b/2 - |\eta|) \tag{5.1}$$

and substituting expression (5.1) into relations (4.1)–(4.3), we obtain the values of the velocities of the fluid motion in the outgoing wave and in the boundary layers on the frictional source. For convenience we will represent the velocity as the superposition of the wave and boundary-layer components $\mathbf{v} = \mathbf{v}^w + \mathbf{v}^b$. Approximate expressions for the components $\mathbf{v}^w$ are written at large distances from the source ($q \gg a, b$), taking into account the smallness of the viscosity ($\delta_N \ll \lambda_c$) and the smallness of the source ($a, b \ll L_\nu$), where $L_\nu = (\nu\Lambda/N)^{1/3}$ is the viscous wave scale. When calculating the boundary-layer velocity $\mathbf{v}^b$ only the low viscosity approximation is used. In addition, it is assumed that $\theta > \varphi$, that is, the wave cone is not crossed by the radiating surface (see the figure). In the opposite case, when part of the wave is reflected from the radiating surface, the solution loses axial symmetry and the expressions become extremely complicated.

Taking into account the solutions of dispersion equations (3.1) with source function (5.1), the asymptotic expressions for velocities of the wave component of the motions (integrals (2.1)) in the accompanying system of coordinates can be written in the form

$$\begin{gathered}(v_x^w, v_y^w, v_z^w) \approx A_1^w(\sin 2\theta\cos\alpha, 2\cos^2\theta\cos\alpha, 2\sin^2\theta)F_1(p, q)\\ A_1^w = -\frac{(i + \operatorname{sign}\mu)(i+1)}{4\sqrt{|\mu|}}\frac{u_0 ab\delta_N}{(2\pi)^{3/2}}\end{gathered} \tag{5.2}$$

The common factor A_1^w is determined solely by the parameters of the radiator (its dimensions and the amplitude of the velocity) and the limiting scale δ_N. The wave function

$$F_1(p, q) = \frac{\cos\varphi\sin\theta - \sin(\pi/4 - \alpha)\sin\varphi\cos\theta}{\sqrt{p\sin\theta + q\cos\theta}}\int_0^{+\infty} k_p^{3/2}\exp\left(ik_p p - \frac{\nu k_p^3 q}{2N\cos\theta}\right)dk_p$$

is a convolution of the solution for a point source [3] and the source function.

Analysis of the form of the wave function shows that, as in the two-dimensional case, a source of small size ($a, b \ll L_\nu$) generates a unimodal beam. The velocity distribution in (5.2) is determined by the trigonometric functions. In this case the axial symmetry of the phase characteristics of the motion (the form of the wave cone) is conserved for all inclinations of the radiating surface if $\theta > \varphi$. However, the wave amplitude varies not only along the generatrix, but also along the direction of the wave cone (the factor $\sin(\pi/4 - \alpha)$). The maximum value of the modulus of the vertical displacements $h_z(0, q)$ in the centre of the unimodal beam is reached when $\alpha = 3\pi/4$ and is

$$h_z(0, q) = \frac{u_0 G ab}{6}\sqrt{\frac{\nu\cos\theta}{2\pi q N}}\left(\frac{2N\cos\theta}{\nu q}\right)^{5/6}\Gamma\left(\frac{5}{6}\right), \quad G = \frac{\operatorname{tg}\theta\sin(\varphi + \theta)}{4\pi^2 N\sqrt{|\mu|}} \tag{5.3}$$

where Γ is the gamma function.

Hence, the displacement velocity of the particles which is proportional to the velocity and area of the source, also depends explicitly on the fluid viscosity, the wave frequency, and the buoyancy frequency. The angular dependence $\sin(\varphi + \theta)$ reflects the geometry of the problem. Formula (5.3), apart from the geometrical factor G and the expression for the area of the radiator, is identical with one of the solutions listed in the table of results presented previously in [4].

The values of the velocities in the boundary layers take the simplest form in the local system of coordinates connected with the radiating surface ($\zeta = 0$):

$$v_x^b \approx u_0 \operatorname{sign}\mu \cos\varphi \exp\left(\frac{i\zeta \operatorname{sign}\mu}{\delta_\varphi} - \frac{\zeta}{\delta_\varphi}\right) - u_0 \frac{(i + \operatorname{sign}\mu)|\mu|^{3/2}}{(i+1)\pi^2 \sin^2\varphi} \delta_N \exp\left(\frac{i\zeta}{\delta_\nu} - \frac{\zeta}{\delta_\nu}\right) W_1$$

$$v_y^b \approx -u_0 \frac{i + \operatorname{sign}\mu}{i+1} \frac{\mu^2 \delta_\varphi^2}{\pi^2} \exp\left(\frac{i\zeta}{\delta_\nu} - \frac{\zeta}{\delta_\nu}\right) W_2 \tag{5.4}$$

$$v_z^b \approx -u_0 \operatorname{sign}\mu \sin\theta \sin\varphi \exp\left(\frac{i\zeta \operatorname{sign}\mu}{\delta_\varphi} - \frac{\zeta}{\delta_\varphi}\right)$$

$$W_1 = \int_{-\infty}^{+\infty} ik_\eta (k_\eta^2 \cos\varphi + k_\xi^2) W_\nu(k_\xi, k_\eta) dk_\xi dk_\eta$$

$$W_2 = \int_{-\infty}^{+\infty} (k_\eta^2 \cos\varphi + k_\xi^2) W_\nu(k_\xi, k_\eta) dk_\xi dk_\eta \tag{5.5}$$

$$W_\nu(k_\xi, k_\eta) = \frac{1}{\chi} \sin\frac{k_\xi a}{2} \sin\frac{k_\eta b}{2} \exp\left\{ik_\xi \xi + ik_\eta \eta + \frac{i-1}{4} \delta_\nu \zeta (k_\xi^2 + k_\eta^2)\right\}$$

The final expressions for the factors W_1 and W_2, describing the dependence of the boundary layer properties on the parameters a, b and δ_ν, calculated by asymptotic methods, are not presented here because of their length.

The spatial structure of the boundary layers is described by the exponential factors which occur in expressions (5.4) and is determined by two transverse scales, characterizing Stokes' periodic flow (δ_ν [5]) and internal boundary flow (δ_φ, as also the solutions presented previously [4]) which contain the common factor δ_N.

The total periodic boundary flow on the source is extremely anisotropic, and the velocity component in the direction of the x axis has the most complex form. Here the motion is characterized by a whole set of microscales, including the universal microscale δ_N. Then $\delta_\nu = f_\nu(\theta)\delta_N$ and $\delta_\varphi = f_\varphi(\varphi, \theta)\delta_N$, and the functions $f_\nu(\theta)$ and $f_\varphi(\varphi, \theta)$ depend only on the source geometry and the normalized wave frequency. In the three-dimensional case under consideration

$$f_\varphi(\theta, \varphi) = \sqrt{2\sin\theta / |\sin^2\theta - \sin^2\varphi|}, \quad f_\nu(\theta) = \sqrt{2/\sin\theta} \tag{5.6}$$

and for an isotropic radiator – a vertical cylinder the expressive for the function $f_\varphi(\varphi, \theta)$ is simplified considerably: $f_\varphi(\varphi, \theta) = \operatorname{tg}\theta$ [4].

All the expressions presented above diverge in the critical case ($\theta = \pm\varphi$), when the wave cone is tangent to one side of the radiating surface, which is a result of using approximate solutions of dispersion equations (3.1). From the structure of the exact expressions it follows that there are no singularities over all range of angular parameters, including the critical case, just as in the problem of the reflection of wave beams [8].

When the radiating surface executes vibrations along the ζ axis being normal to the boundary, i.e.

$$\mathbf{u} = u\mathbf{e}_\zeta \tag{5.7}$$

a radiator that excites waves both in a viscous fluid and in an ideal stratified fluid is obtained (a source of the piston type or of variable flow rate). Values of wave components of the velocity are found by substituting expression (5.7) into relation (4.1). We obtain

$$(v_x^w, v_y^w, v_z^w) \approx A_2^w(\sin 2\theta \cos\alpha, 2\cos^2\theta\cos\alpha, 2\sin^2\theta)F_2(p, q)$$
$$A_2^w = -\frac{(1-i)}{8\pi^{3/2}}\frac{u_0 ab}{\sqrt{\sin\theta}} \tag{5.8}$$

The factor A_2^w depends only on the source parameters (the dimensions, velocity, and normalized frequency), while the wave function has the form

$$F_2(p, q) = \frac{1}{\sqrt{p\sin\theta + q\cos\theta}}\int_0^{+\infty} k_p^{1/2}\exp\left(ik_p p - \frac{\nu k_p^3 q}{2N\cos\theta}\right)dk_p$$

In this case the vertical displacements, as well as the wave phase, are axially symmetric for an arbitrary inclination of the radiating surface ($\theta > \varphi$). The maximum value of the modulus of the vertical displacements in the centre of the beam

$$h_z(0, q) \approx \frac{u_0 ab}{12\pi}\sqrt{\frac{\sin\theta}{\nu N}}\frac{1}{q} \tag{5.9}$$

increases as the area and velocity of the source increase and decreases in inverse proportion to the distance.

In a similar manner one can calculate expressions for exponentially decreasing boundary layers, the spatial structure of which in an arbitrary case is characterized by two scales δ_ν and δ_φ (as for the frictional source). We obtain

$$v_x^b \approx -\frac{u_0}{\pi^2}\operatorname{sign}\mu\cos\varphi\exp\left(\frac{i\zeta\operatorname{sign}\mu}{\delta_\varphi} - \frac{\zeta}{\delta_\varphi}\right)W_6 - \frac{(1+i)u_0\operatorname{sign}\mu}{\sqrt{2}\,\pi^2\sin\varphi}\exp\left(\frac{i-1}{\delta_\nu}\zeta\right)W_7$$
$$v_y^b \approx -\frac{u_0}{\pi^2}\operatorname{sign}\mu\sqrt{\sin\theta}\exp\left(\frac{i-1}{\delta_\nu}\zeta\right)W_8 - \frac{(1+\operatorname{sign}\mu)u_0\delta_\varphi\operatorname{ctg}\varphi}{2\pi^2}\exp\left(\frac{i\operatorname{sign}\mu - 1}{\delta_\varphi}\zeta\right)W_9 \tag{5.10}$$
$$v_z^b \approx \frac{u_0}{\pi^2}\operatorname{sign}\mu\sin\varphi\exp\left(\frac{i\zeta\operatorname{sign}\mu}{\delta_\varphi} - \frac{\zeta}{\delta_\varphi}\right)W_{10}$$

The expressions for the standard functions $W_6 - W_{10}$ of the type (5.5) are not presented because of their complexity.

On the line of maximum displacements ($\alpha = 3\pi/4$) on the wave cone with vertex angle $\theta = \arccos(\omega/N)$ the wave amplitudes for the sources under consideration (5.3) and (5.9) become equal at a distance

$$q_0 = \frac{\sin^3(\varphi+\theta)\sqrt{\sin\theta}\operatorname{tg}\theta}{(2\pi)^{9/2}|\mu|}\delta_\varphi\Gamma^3\left(\frac{5}{6}\right) \tag{5.11}$$

At short distances, in fact in the boundary-layer domain or in the transition range ($q < q_0$), when the contribution from the factor characterizing the geometry of the radiator is predominant, the frictional radiator turns out to be more effective. At long distances from the radiator, namely, in the wave domain ($q > q_0$), the pattern of disturbances is governed by the geometrical decay, and a source of the piston type proves to be more effective, in agreement with energy estimates [9].

6. CONCLUSION

Following to the method proposed previously [4], a complete solution of the three-dimensional problem of the generation of disturbances in a viscous continuously stratified fluid by part of an inclined plane that executes periodic motions in an arbitrary direction has been constructed. This solution describes motions of different scales, namely, three-dimensional beams of internal waves and two types of boundary layers. The thickness of one of them, which is an analog of Stokes' periodic motion in a homogeneous fluid, is determined by the kinematic viscosity and the wave frequency $\delta_\varphi = f_\varphi(0)\delta_N$. The thickness of the other layer, namely, the internal boundary layer, that has no analog in the theory of a homogeneous fluid, additionally depends on the geometry of the problem via the parameter $\delta_\varphi = f_\varphi(\varphi, \theta)\delta_N$. The universal microscale $\delta_N = \sqrt{\nu/N}$ that occurs in these expressions is determined by the kinematic viscosity and the buoyancy frequency of the medium. The trigonometric function $f_\varphi(\theta, \varphi)$ has been calculated for an inclined plane (the first expression in (5.6)) and for a vertical cylinder [4] ($f_\varphi(\varphi, \theta) = \text{tg}\,\theta$). For a viscous homogeneous fluid the viscous and internal boundary layers merge into a single degenerate boundary layer.

Due to the interconnection between all the elements of the family of periodic motions in a continuously stratified fluid the decay time of singular components (the boundary layers on a rigid surfaces and internal boundary flows on the surfaces of discontinuity of the density and its derivatives in the bulk of the fluid) is determined by the time for which the whole set of motions exists, that is, by the decay period of large-scale waves.

In the important practical case of exponential stratification and in the low viscosity approximation an asymptotic estimate of the exact solution was made for two types of sources – frictional and piston. Expressions (5.2), (5.4), (5.8) and (5.10), which describe the wave beams and two families of boundary layers, satisfy the boundary conditions of the problem exactly, do not contain any additional adjusting parameters and can be directly used for comparison with laboratory experiments and natural observations. In limiting cases the solutions obtained reduce continuously to the previously constructed solutions of the two-dimensional problem [4]. When molecular effects are taken into account the structure of the boundary layers becomes more complex due to the formation of flows induced by diffusion [9] and due to splitting of the diffusion and velocity boundary layers when the values of the kinetic coefficients, namely, the kinematic viscosity and the diffusivity of the stratifying components, are not the same. When solving non-linear equations one must take into account the interaction of all the elements of the periodic flows, i.e. both the internal waves and all types of boundary flows also.

We wish to dedicate this paper to the memory of Yu. V. Kistovich who died after a short serious illness on December 27, 2001.

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